ODEs. Integrating Factors. Test for exactness. If exact, solve. If not, use an integrating factor as given or obtained by inspection or by the theorems in the text. Also, if an initial condition is given, find the corresponding particular solution.

1. $2\,xy\,dx + x^2\,dy = 0$

```
ClearAll["Global`*"]
```

```
eqn = 2 x y[x] + x^2 y'[x] == 0;
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, \frac{C[1]}{x^2}]\}\}$

```
eqn /. sol
```

```
{True}
```

```
ClearAll["Global`*"]
```

2. $x^3 + y[x]^3\, y'[x] = 0$

```
eqn = x^3 + y[x]^3 y'[x] == 0;
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, -(-x^4 + 4\,C[1])^{1/4}]\},$
$\{y \to \text{Function}[\{x\}, -i\,(-x^4 + 4\,C[1])^{1/4}]\},$
$\{y \to \text{Function}[\{x\}, i\,(-x^4 + 4\,C[1])^{1/4}]\},$
$\{y \to \text{Function}[\{x\}, (-x^4 + 4\,C[1])^{1/4}]\}\}$

```
eqn /. sol[[1]]
```

```
True
```

```
eqn /. sol[[2]]
```

```
True
```

```
eqn /. sol[[3]]
```

```
True
```

```
eqn /. sol[[4]]
```

```
True
```

3. $\sin x \cos y + \cos x \sin yy' = 0$

```
ClearAll["Global`*"]
```

```
eqn = Sin[x] Cos[y[x]] + Cos[x] Sin[y[x]] y'[x] == 0;
sol = DSolve[eqn, y, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{y \to \text{Function}\left[\{x\}, -\text{ArcCos}\left[\frac{1}{2} C[1]\ \text{Sec}[x]\right]\right]\right\}, \left\{y \to \text{Function}\left[\{x\}, \text{ArcCos}\left[\frac{1}{2} C[1]\ \text{Sec}[x]\right]\right]\right\}\right\}$$

Though equivalent to the green cell above, the text answer is expressed as $\arccos(\frac{c}{\cos x})$.

```
eqn /. sol[[1]]
True

eqn /. sol[[2]]
True
```

4. $e^{3\theta}(r'[\theta] + 3\, r[\theta]) = 0$

```
ClearAll["Global`*"]

eqn = ⅇ^(3 θ) (r'[θ] + 3 r[θ]) == 0;
sol = DSolve[eqn, r, θ]
```

$$\{\{r \to \text{Function}[\{\theta\}, e^{-3\theta}\, C[1]]\}\}$$

```
eqn /. sol
{True}
```

5. $(x^2 + y^2) - 2\,xyy' = 0$

```
ClearAll["Global`*"]

eqn = x^2 + y[x]^2 - 2 x y[x] y'[x] == 0;
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, -\sqrt{x}\ \sqrt{x + C[1]}\right]\right\}, \left\{y \to \text{Function}\left[\{x\}, \sqrt{x}\ \sqrt{x + C[1]}\right]\right\}\right\}$$

```
Simplify[eqn /. sol[[1]]]
True

Simplify[eqn /. sol[[2]]]
True
```

6. $3\,(y + 1) = 2\,xy',\ \ (y + 1)\,x^{-4}$

```
ClearAll["Global`*"]

eqn = 3 (y[x] + 1) == 2 x y'[x];
sol = DSolve[eqn, y, x]

{{y → Function[{x}, -1 + x^(3/2) C[1]]}}

eqn /. sol

{True}
```

7. $2x\tan y + \sec^2 y\, y' = 0$

```
ClearAll["Global`*"]

eqn = 2 x Tan[y[x]] + Sec[y[x]]^2 y'[x] == 0;
sol = DSolve[eqn, y, x]

Solve::ifun :
  Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

{{y → Function[{x}, ArcCot[e^(x^2 - 2 C[1])]]}}

Simplify[eqn /. sol]

{True}
```

The Mathematica solution checks out above. I believe equivalent is the text answer y=$e^{x^2}\tan y = c$.

8. $e^x(\cos y - \sin y\, y') = 0$

```
ClearAll["Global`*"]

eqn = e^x (Cos[y[x]] - Sin[y[x]] y'[x]) == 0;
sol = DSolve[eqn, y, x]

Solve::ifun :
  Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

{{y → Function[{x}, -ArcCos[e^(-x - C[1])]]},
 {y → Function[{x}, ArcCos[e^(-x - C[1])]]}}

Simplify[eqn /. sol[[1]]]

True

Simplify[eqn /. sol[[2]]]

True
```

9. $e^{2x}(2\cos y - \sin y\, y') = 0, \;\; y(0) = 0$

```
ClearAll["Global`*"]
```

```
eqn = ⅇ^(2 x) (2 Cos[y[x]] - Sin[y[x]] y'[x]) == 0;
sol = DSolve[{eqn, y[0] == 0}, y, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

General::stop: Further output of Solve::ifun will be suppressed during this calculation. >>

```
{{y → Function[{x}, -ArcCos[ⅇ^(-2 x)]]}, {y → Function[{x}, ArcCos[ⅇ^(-2 x)]]}}
```

```
Simplify[eqn /. sol[[1]]]
True
```

```
Simplify[eqn /. sol[[2]]]
True
```

The Mathematica solution checks out above. I believe it is equivalent is the text answer $e^{x^2} \cos y = 1$.

11. $2 \cosh x \cos y = \sinh x \sin y\, y'$

```
In[4]:= ClearAll["Global`*"]
```

```
In[7]:= eqn = 2 Cosh[x] Cos[y[x]] == Sinh[x] Sin[y[x]] y'[x];
        sol = DSolve[eqn, y, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
Out[8]= {{y → Function[{x}, -ArcCos[-1/2 ⅈ C[1] Csch[x]^2]]},
         {y → Function[{x}, ArcCos[-1/2 ⅈ C[1] Csch[x]^2]]}}
```

```
In[9]:= Simplify[eqn /. sol[[1]]]
Out[9]= True
```

```
In[10]:= Simplify[eqn /. sol[[2]]]
Out[10]= True
```

```
In[12]:= Solve[-ArcCos[-1/2 ⅈ C[1] Csch[x]^2] == y[x], C[1]]
Out[12]= {{C[1] → ConditionalExpression[
           2 ⅈ Cos[y[x]] Sinh[x]^2, (-Re[y[x]] == 0 && -Im[y[x]] ≥ 0) ||
            0 < -Re[y[x]] < π || (-Re[y[x]] == π && -Im[y[x]] ≤ 0)]}}
```

The cell above looks pretty close to the text answer. However, since, unless directed otherwise, I take the constant c a being real, I can't call the answers equivalent. (The text answer is c=sinh^2 x cos y). It does appear that the Mathematica solutions are effective.

12. $(2\,xy + y')\,e^{x^2} = 0,\ y(0) = 2$

```
ClearAll["Global`*"]
```

```
eqn = (2 x y[x] + y'[x]) e^(x^2) == 0;
sol = DSolve[{eqn, y[0] == 2}, y, x]
```

$\{\{y \to \text{Function}[\{x\},\ 2\,e^{-x^2}]\}\}$

```
eqn /. sol
```

```
{True}
```

13. $e^{-y[x]} + e^{-x}\left(-e^{-y[x]} + 1\right) y'[x] = 0,\ F = e^{x+y[x]}$

```
ClearAll["Global`*"]
```

```
eqn = e^(-y[x]) + e^(-x) (-e^(-y[x]) + 1) y'[x] == 0;
sol = DSolve[eqn, y, x]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$\{\{y \to \text{Function}[\{x\},\ e^x - C[1] - \text{ProductLog}[-e^{e^x - C[1]}]]\}\}$

```
Simplify[eqn /. sol]
```

```
{True}
```

In[20]:= $\text{Solve}[y[x] + \text{ProductLog}[-e^{e^x - C[1]}] - e^x == -C[1],\ C[1]]$

Out[20]= $\{\{C[1] \to e^x + e^{y[x]} - y[x]\}\}$

In[21]:= $\text{PossibleZeroQ}[(e^x + e^{y[x]} - y[x]) - (e^x - y[x] + e^{y[x]})]$

Out[21]= True

The green cell above agrees with the text answer, as shown by the PZQ. According to *MathWorld*, LambertW[k, z] autoevaluates to ProductLog[k, z] in the Wolfram Language.

14. $(a + 1)\,y + (b + 1)\,x\,y' = 0,\ y(1) = 1,\ F = x^a\,y^b$

```
ClearAll["Global`*"]
```

```
eqn = (a + 1) y[x] + (b + 1) x y'[x] == 0;
sol = DSolve[{eqn, y[1] == 1}, y, x]
```

$\{\{y \to \text{Function}[\{x\},\ (1 + b)^{\frac{1}{1+b} + \frac{a}{1+b}}\,(x + b\,x)^{-\frac{1}{1+b} - \frac{a}{1+b}}]\}\}$

```
Simplify[eqn /. sol]
{True}
```

15. Exactness. Under what conditions for the constants a, b, k, l is (a x + b y)dx + (k x + l y)dy = 0 exact? Solve the exact ODE.

```
ClearAll["Global`*"]
```

According to the exactness test, b = k.

The text answer also has the relationship a*x^2 + 2*k*x*y + l*y^2 = c, but I haven't been able to track this down yet. As for the exact equation, (and substituting b for k)

eqn = $y'[x] == -\dfrac{(a\,x + b\,y[x])}{(b\,x + l\,y[x])}$

$$y'[x] == -\frac{a\,x + b\,y[x]}{b\,x + l\,y[x]}$$

```
sol = DSolve[eqn, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{-b\,x - \sqrt{e^{2\,C[1]}\,l + b^2\,x^2 - a\,l\,x^2}}{l}\right]\right\}, \left\{y \to \text{Function}\left[\{x\}, \frac{-b\,x + \sqrt{e^{2\,C[1]}\,l + b^2\,x^2 - a\,l\,x^2}}{l}\right]\right\}\right\}$$

```
FullSimplify[eqn /. sol[[1]]]
True
```

```
FullSimplify[eqn /. sol[[2]]]
True
```